

COHOMOLOGY RINGS AND FORMALITY PROPERTIES OF NILPOTENT GROUPS

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ABSTRACT. We introduce partial formality and relate resonance with partial formality properties. For instance, we show that for finitely generated nilpotent groups that are k -formal, the resonance varieties are trivial up to degree k . We also show that the cohomology ring of a nilpotent k -formal group is generated in degree 1, up to degree $k+1$; this criterion is necessary and sufficient for 2-step nilpotent groups to be k -formal. We compute resonance varieties for Heisenberg-type groups and deduce the degree of partial formality for this class of groups.

CONTENTS

1. Introduction and statement of results	1
2. Partial minimal models and formality properties	3
3. Obstructions to partial formality	6
4. Partial formality and resonance	9
5. Heisenberg-type groups	13
References	17

1. INTRODUCTION AND STATEMENT OF RESULTS

A space which has the minimal model isomorphic to the minimal model of its cohomology ring is called formal. In other words, the "rational homotopy type" of

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the space is a formal consequence of its cohomology ring. Compact Kähler manifolds (in particular, smooth complex projective varieties) are important examples of formal spaces. See Deligne-Griffiths-Morgan-Sullivan [5] for details.

It seems natural to introduce a more relaxed version, namely *k-formality* (see Definition 2.5), following [5]. For $k = 1$, it coincides with the usual notion of 1-formality present in the literature (see [1]). See [1] for the equivalence between 1-formality of a space and quadratic presentability of the Malcev Lie algebra associated to the fundamental group of that space, and also [3].

Our notion of partial formality is strictly weaker, despite the terminology, than the one introduced by Fernández-Muñoz in [10, Definition 2.2]. For instance, in the case of nilmanifolds, formality is equivalent to 1-formality in the sense of [10] (see [10, Lemma 2.6]), which is not the case with our definition (see Corollary 5.9). This is due to the fact that the test of partial formality in [10] is global, in the sense that it involves the whole minimal model, whereas our *k-formality* test is a finite one, using only information provided by the *k*-minimal model, up to degree $k + 1$; see Proposition 3.1(1), and the discussion following it.

It is well-known that 1-formality (in our sense) is the first general obstruction in the Serre problem regarding the characterization of *projective groups* (fundamental groups of smooth projective complex varieties). A difficult particular case of this problem turns out to be the one of nilpotent groups. A positive answer is given by Campana [2] for a certain class of 2-step nilpotent groups, the Heisenberg groups $\mathcal{H}_{n \geq 4}$ (see Definition 5.1). As for the remaining Heisenberg groups, \mathcal{H}_1 does not pass the 1-formality criterion (see Corollary 5.9 for a slightly more general result); \mathcal{H}_2 and \mathcal{H}_3 are also non-projective groups (see Carlson-Toledo [3, Corollary 4.5]). When passing to 3-step nilpotent groups, the answer is not known, according to [3].

We approach in Corollary 5.10 the solution to the Serre problem given by Campana, to point out some homotopic features of the projective smooth complex varieties constructed in [2]. This is actually a consequence of a more general result (where partial formality for a group G is defined via the classifying space $K(G, 1)$).

Theorem 1.1. *Let M be a k -formal space such that $\pi_1(M)$ is not k -formal ($k \geq 2$). Then there exists $2 \leq i \leq k$ such that $\pi_i(M) \neq 0$.*

Note that the above result no longer holds if partial formality is taken in the sense of [10]; see Remark 2.8.

Some of the results presented in Section 3, such as passing from partial to full formality (Proposition 3.6) and Proposition 3.1(2) are inspired by the similar results obtained in the 1-connected case by Papadima in [18].

For nilpotent groups we find obstructions to (partial) formality involving either generators of the cohomology ring (up to a degree) or certain *resonance varieties* (see Definition 4.1) associated to the cohomology ring.

Theorem 1.2. *Let G be a finitely generated nilpotent group.*

- (1) *If G is k -formal, then $H^{\leq k+1}(G)$ is generated as an algebra by $H^1(G)$.*
- (2) *For 2-step nilpotent groups, the converse also holds.*

For $k = 1$, the first part of Theorem 1.2 follows from [1], Lemma 3.17 on page 35. The second part of Theorem 1.2, for the case $k = 1$, follows from [3], Corollary 0.2.1. As explained in Remark 4.7, the second part may fail, if G is not 2-step nilpotent, even for $k = 1$.

Another obstruction to partial formality can be expressed in terms of the resonance varieties $\mathcal{R}_1^i(G) \subseteq H^1(G, \mathbb{k})$, over a field \mathbb{k} of characteristic zero.

Theorem 1.3. *Let G be a nilpotent, finitely generated, s -formal group. Then the resonance varieties of G are trivial up to degree s , that is, $\mathcal{R}_1^i(G) \subseteq \{0\}$ for $i \leq s$.*

Note that this result, via our Lemma 4.5, generalizes [3, Lemma 2.4], corresponding to the case $s = 1$.

For fundamental groups of complements of arrangements of complex hyperplanes, it is well-known that 1-formality holds. It turns out that our nilpotency test from Theorem 1.3, via resonance, is faithful (see Example 4.3).

In Section 5 we make an analysis of the formality properties for Heisenberg-type groups from a double perspective - generators of the cohomology ring and resonance varieties.

2. PARTIAL MINIMAL MODELS AND FORMALITY PROPERTIES

For D. Sullivan's theory of minimal models we refer to [19] (see also [5], [9], [11] and [16]).

Let (A^*, d_A) be a differential graded algebra (D.G.A.) over a field \mathbb{k} of characteristic zero, such that $H^0(A^*, d_A)$ is the ground field. A minimal model for A^* is a minimal D.G.A. $(\mathcal{M}, d_{\mathcal{M}})$ such that there exists a morphism of D.G.A.'s $\rho : \mathcal{M} \rightarrow A^*$ inducing isomorphism on cohomology. There is a unique (up to isomorphism) $\mathcal{M} = \mathcal{M}(A)$ satisfying the conditions in the definition.

Let K be a space having the homotopy type of a connected simplicial complex. We call the minimal model of K , denoted $\mathcal{M}(K)$, the minimal model associated to the D.G.A. of p.l. forms $\Omega^*(K)$.

A D.G.A. A^* as above is called formal if there exists a D.G.A. morphism $(\mathcal{M}(A), d_{\mathcal{M}}) \rightarrow (H^*(A), d = 0)$ which induces isomorphism in cohomology.

K is formal if and only if the minimal model of K is a formal D.G.A, i.e. $\mathcal{M}(K) = \mathcal{M}(H^*(K), d = 0)$.

Definition 2.1. A minimal algebra \mathcal{M} generated by elements of degree $\leq k$ is called a k -minimal model of a D.G.A. (A^*, d_A) if there exists a D.G.A. map $\rho : \mathcal{M} \rightarrow A$ such that it induces in cohomology isomorphisms up to degree k and a monomorphism in degree $k + 1$. Again, such an object exists and is uniquely determined, up to isomorphism, for any D.G.A. (A^*, d_A) . Notation: $\mathcal{M} = \mathcal{M}_k(A)$.

Remark 2.2. If $\mathcal{M} = (\wedge V, d)$ is a minimal algebra, then $\mathcal{M}_k(\mathcal{M}) = (\wedge V^{\leq k}, d)$. For a space K , set $\mathcal{M}_k(K) := \mathcal{M}_k(\mathcal{M}(K))$.

Example 2.3. Assume $H^*(K) = \wedge(x_1, \dots, x_n)$, as rings. Then the minimal model of K is $\mathcal{M}(K) = (\wedge(x_1, \dots, x_n), d = 0)$, hence K is formal. If $\deg(x_i) = 1$, for all i , then $\mathcal{M}_1(K) = \mathcal{M}(K)$; it follows from [19] that the rational associated graded Lie algebra of $G = \pi_1(K)$, $\text{gr}(G) \otimes \mathbb{Q}$, is abelian, concentrated in degree 1 (i.e. $\text{gr}(G)_{\geq 2} \otimes \mathbb{Q} = 0$).

Definition 2.4. A D.G.A. (A^*, d_A) is called k -formal if there exists a D.G.A. morphism $(\mathcal{M}_k(A), d) \rightarrow (H^*(A), 0)$ which induces isomorphisms in cohomology up to degree k and a monomorphism in degree $k + 1$.

Definition 2.5. The space K is called k -formal if $\mathcal{M}(K)$ is a k -formal D.G.A. In other words, $\mathcal{M}_k(K) = \mathcal{M}_k(H^*(K), 0)$.

A group G is called formal (respectively k -formal) if the associated Eilenberg-MacLane space $K(G, 1)$ is formal (respectively k -formal). This convention (replace the $K(G, 1)$ space by the group G) will be used from now on.

Note that a formal space is k -formal, for any k . A partial converse will be proved later, in Proposition 3.6.

Recall from [21] that a continuous map between connected CW -complexes $f : X \rightarrow Y$ is called a k -homotopy equivalence if it induces isomorphisms on homotopy groups up to degree $k - 1$ and a surjection in degree k . Up to homotopy, we can see f as an inclusion. Then from the long exact homotopy sequence associated to the pair (Y, X) we get $\pi_{\leq k}(Y, X) = 0$, hence, by the Hurewicz theorem (relative version), $H_{\leq k}(Y, X) = 0$. It follows that $H^{\leq k}(Y, X) = 0$, and we

can apply the long exact cohomology sequence of the pair to conclude that we have $H^{\leq k-1}(X) \cong H^{\leq k-1}(Y)$ and an injection $H^k(Y) \hookrightarrow H^k(X)$, both induced by the inclusion f . Therefore a k -homotopy equivalence is a *homology k -equivalence* (that is, a map which induces isomorphisms on cohomology groups up to degree $k-1$ and a monomorphism in degree k).

Proposition 2.6. *Let $f : X \longrightarrow Y$ be a homology k -equivalence. Then*

$$(2.1) \quad \mathcal{M}_{k-1}(X) \cong \mathcal{M}_{k-1}(Y)$$

and

$$(2.2) \quad \mathcal{M}_{k-1}(H^*(X), 0) \cong \mathcal{M}_{k-1}(H^*(Y), 0)$$

Proof. Let $\rho : \mathcal{M}_{k-1}(Y) \longrightarrow \Omega^*(Y)$ be a map as in the definition of the k -minimal model. Since f is a homology k -equivalence, the map $f^* \circ \rho : \mathcal{M}_{k-1}(Y) \longrightarrow \Omega^*(X)$ satisfies the conditions from the definition of the k -minimal model, so $\mathcal{M}_{k-1}(Y)$ is also the $(k-1)$ -minimal model of X . A similar proof shows that $\mathcal{M}_{k-1}(H^*(X), 0) \cong \mathcal{M}_{k-1}(H^*(Y), 0)$. \square

Corollary 2.7. *Assume f is a homology k -equivalence. Then X is $(k-1)$ -formal if and only if Y is $(k-1)$ -formal.*

Remark 2.8. The previous result is similar to [10, Theorem 5.2(i)], where only “ Y $(k-1)$ -formal $\Rightarrow X$ $(k-1)$ -formal” is shown, but uses a different notion of partial formality, as explained in Remark 5.3. With the definition of partial formality in the sense of [10], Theorem 1.1 no longer holds. This can be seen by considering M to be the projective smooth complex variety with fundamental group a Heisenberg group $\mathcal{H}_{n \geq 4}$ (see Definition 5.1), constructed by Campana ([2]). Then M is formal, hence k -formal in the sense of [10, Definition 2.2], for any k , but $\pi_1(M)$ is not even 1-formal in the sense of [10] (by [10, Lemma 2.6]). Were Theorem 1.1 true for $k=2$ it would imply that $\pi_2(M) \neq 0$. One can deduce that $\pi_2(M) = 0$ from the construction of M (see [2], or [3, Section 5]), when $n \geq 6$.

Theorem 1.1 follows from the result below:

Theorem 2.9. *Assume either M is a k -formal space such that $\pi_1(M)$ is not k -formal or M is not k -formal and $\pi_1(M)$ is k -formal, where $k \geq 2$. Then there exists $2 \leq i \leq k$ such that $\pi_i(M) \neq 0$.*

Proof. Assume $\pi_i(M) = 0$, $\forall 2 \leq i \leq k$ and set $\pi_1(M) := H$. Consider the classifying map $f : M \longrightarrow K(H, 1)$ such that $\pi_1(f) = \text{id}_H$. This map is then a

$(k+1)$ -homotopy equivalence. Then M is k -formal if and only if H is k -formal, by Corollary 2.7, contradicting our hypothesis. \square

3. OBSTRUCTIONS TO PARTIAL FORMALITY

We begin by giving an alternative characterization of (partial) formality, along the lines from [5] and [10].

We will need several basic properties of the bigraded minimal model of a connected, graded-commutative algebra H^* , extracted from [11].

As an algebra, $\mathcal{B} = \wedge Z$, where Z is bigraded by $Z = \bigoplus_{i \geq 0, p \geq 1} Z_i^p$. The differential d has degree $+1$ with respect to upper degrees and is of degree -1 with respect to lower degrees. Moreover,

$$(3.1) \quad H_+(\mathcal{B}, d) = 0$$

The k -minimal model of \mathcal{B} will be denoted by ${}_k\mathcal{B} := (\wedge Z^{\leq k}, d)$.

For a D.G.A. $(\wedge V, d)$, set $C^* = \text{Ker}(d|_{V^*})$.

Proposition 3.1. *The following hold.*

- (1) A D.G.A. (A^*, d_A) is k -formal ($1 \leq k \leq \infty$) if and only if it has a k -minimal model $\mathcal{M}_k = (\wedge V^{\leq k}, d)$ with a decomposition $V^{\leq k} = C^{\leq k} \oplus N^{\leq k}$ such that $(N \cdot \mathcal{M}_k \cap \text{Ker } d)^{\leq k+1} \subset d\mathcal{M}_k$, where $N = \bigoplus_{1 \leq i \leq k} N^i$.
- (2) Assume $\mathcal{M}_k = (\wedge V^{\leq k}, d)$ has the property from Part (1) above. Let $\phi : \wedge C^{\leq k} \longrightarrow H^*(\mathcal{M}_k)$ be the map of graded algebras which associates to an element in $\wedge C^{\leq k}$ its cohomology class. Then ϕ is surjective up to degree $k+1$.
- (3) If M is a k -formal space and a rational $K(\pi, 1)$, then the cohomology algebra of M is generated by $H^1(M)$, up to degree $k+1$, i.e. $H^{\leq k+1}(M) = (H^1(M))^{\leq k+1}$.

Proof. (1) If A^* is k -formal, then there is a D.G.A. isomorphism $\mathcal{M}_k(A) \cong {}_k\mathcal{B}$, where \mathcal{B} is the bigraded model of $H^*(A)$. The required decomposition of $Z^{\leq k}$ is as follows: $C^q = Z_0^q$ and $N^q = \bigoplus_{i > 0} Z_i^q$, for $q \leq k$. Let $x \in N \cdot {}_k\mathcal{B} \cap \text{Ker } d$, be homogeneous of upper degree $q \leq k+1$. The lower degree of each component of x is strictly positive, hence $x = d(z)$, for some $z \in \mathcal{B}^{q-1} = {}_k\mathcal{B}^{q-1}$, since $H_+(\mathcal{B}) = 0$ and $q \leq k+1$.

To prove the converse claim, define a G.A. map

$$(3.2) \quad \rho : \mathcal{M}_k \longrightarrow H^*(\mathcal{M}_k)$$

by $c \mapsto [c]$ and $n \mapsto 0$, for $c \in C^{\leq k}$, $n \in N^{\leq k}$.

We begin by showing that ρ is in fact a D.G.A. map, that is $\rho(d(v)) = 0$, for any $v \in V^{\leq k}$.

For $c \in C^{\leq k}$ this is true, since $d(c) = 0$.

Take $n \in N^{\leq k}$. We can write $d(n) = \bar{n} + \bar{c}$, $\bar{n} \in N \cdot \mathcal{M}_k$, $\bar{c} \in \wedge C^{\leq k}$. Then $\bar{n} \in (N \cdot \mathcal{M}_k \cap \text{Ker } d)^{\leq k+1}$, so \bar{n} is a boundary in \mathcal{M}_k , which implies $\rho(d(n)) = [\bar{c}] = 0$.

Let us prove now the injectivity of the map $H^q(\rho)$, for $q \leq k+1$. Take $\alpha \in \mathcal{M}_k = \wedge V^{\leq k}$ homogeneous of degree q , such that $H^q(\rho)([\alpha]) = 0$ and write α as a sum $\alpha = \alpha_1 + \alpha_2$, where $\alpha_1 \in \wedge C^{\leq k}$ and $\alpha_2 \in N \cdot \mathcal{M}_k$. It follows that $d(\alpha_2) = d(\alpha - \alpha_1) = 0$, hence $\alpha_2 = d(z)$, $z \in \mathcal{M}_k$, so $[\alpha] = [\alpha_1]$. This last equality, together with $H^q(\rho)([\alpha]) = 0$ implies $[\alpha] = [\alpha_1] = 0$, hence $H^q(\rho)$ is injective.

We prove the surjectivity of $H^q(\rho)$, for $q \leq k+1$. Let $[\alpha] \in H^q(\mathcal{M}_k)$, with α written just as before as a sum $\alpha = \alpha_1 + \alpha_2$. Again α_2 is exact in \mathcal{M}_k , and $H^q(\rho)([\alpha_1]) = [\alpha]$, hence surjectivity is proven.

To end the proof, consider the composition of ρ with the map induced in cohomology by the map in Definition 2.1.

(2) The proof of the fact that ϕ is surjective up to degree $k+1$ goes the same way as the proof of the surjectivity of $H^q(\rho)$ in Part (1).

(3) We infer from k -formality the existence of a k -minimal model of M , \mathcal{M}_k , having the property from Part (1). Since M is a rational $K(\pi, 1)$, $V^{\leq k} = V^1$ and $C^{\leq k} = C^1$. Moreover, \mathcal{M}_k is actually the minimal model of M . Our claim follows by considering the map $\phi : \wedge C^1 \longrightarrow H^*(M)$ defined in Part (2). \square

Proposition 3.1(1) shows that our notion of k -formality from Definition 2.4 is less restrictive than the one from [10, Definition 2.2]. Actually, the requirements in [10] are strictly stronger than ours; see Remark 5.3. In 3.1(2) we assume less than in [10], and we obtain more.

Definition 3.2. A 2-step nilpotent group is a group G such that $[G, [G, G]] = 0$, where $[\cdot, \cdot]$ denotes the group commutator.

Remark 3.3. To say that G is a finitely generated 2-step nilpotent group is equivalent to say that G is a central extension of some finite rank abelian group by another finite rank abelian group.

Proof of Theorem 1.2 (1) Follows from Proposition 3.1 Part (3), due to the fact that $K(G, 1)$ is a rational $K(\pi, 1)$.

(2) Since G is two-step nilpotent, we can choose the generators of the minimal model such that $\mathcal{M}(G) = \wedge(x_1, \dots, x_m) \otimes \wedge(y_1, \dots, y_n)$, $\deg(x_i) = \deg(y_j) = 1$, $d(y_j) \in \wedge^2(x_1, \dots, x_m)$, $\forall j$ and $C^1 = \langle x_1, \dots, x_m \rangle$.

Define then a map ϕ (see Definition 2.4), $\phi : (\mathcal{M}, d) \longrightarrow (H^*(G), 0)$ by $x_i \mapsto [x_i]$ and $y_j \mapsto 0$. It is easy to see that ϕ is a D.G.A. morphism and $H^1(\phi)$ is the identity. As a consequence of the fact that $H^{\leq k+1}(G) = (H^1(G))^{\leq k+1}$, $H^{\leq k+1}(\phi)$ is also the identity, being induced by $H^1(\phi)$. This proves the k -formality of G . \square

Corollary 3.4 ([12]). *If G is a finitely generated nilpotent group, then G is formal if and only if it is rationally abelian.*

Proof. To justify the less obvious implication, assume the minimal model of the finitely generated nilpotent group G is $\mathcal{M} = \wedge(x_1, \dots, x_n, y_{n+1}, \dots, y_p)$, with $C^1 = \langle x_1, \dots, x_n \rangle$. One knows that $\dim H^p(\mathcal{M}) = 1$, since \mathcal{M} has the same cohomology as a p -dimensional nilmanifold. On the other hand, from Theorem 1.2, Part (1), for $k = \infty$, we get that $H^*(\mathcal{M})$ is a quotient of the algebra $\wedge^*(x_1, \dots, x_n)$, hence $p = n$, which proves our claim. \square

We will also need the following lemma:

Lemma 3.5. *Let (A^*, d_A) be a D.G. algebra. Any k -minimal model $\mathcal{M}_k = (\wedge V^{\leq k}, d) \xrightarrow{\phi_k} (A^*, d_A)$ can be extended to a $(k+1)$ -minimal model $\mathcal{M}_{k+1} = (\wedge V^{\leq k+1}, d) \xrightarrow{\phi_{k+1}} (A^*, d_A)$ such that for any $v \in V^{k+1}$, $d(v) \in d(\mathcal{M}_k)$ if and only if $d(v) = 0$. Moreover, $V^{k+1} = \bigoplus_{i \geq 0} V_i^{k+1}$ and $d(v) = 0$ if and only if $v \in V_0^{k+1}$.*

Proof. Following a standard technique we gradually define the vector space of degree $k+1$ generators V^{k+1} .

Set $V_0^{k+1} := \text{coker } H^{k+1}(\phi_k)$, with $d|_{V_0^{k+1}} = 0$. Assume we have already defined V_i^{k+1} such that there is an extension $\phi_{ki} : \wedge(V^{\leq k} \oplus V_{\leq i}^{k+1}) \rightarrow (A^*, d_A)$ of the morphism ϕ_k and set $V_{i+1}^{k+1} := \text{Ker } H^{k+2}(\phi_{ki})$ with transgression given by the inclusion $[d] : V_{i+1}^{k+1} \hookrightarrow H^{k+2}(\wedge(V^{\leq k} \oplus V_{\leq i}^{k+1}))$. Define $V^{k+1} = \bigoplus_{i \geq 0} V_i^{k+1}$.

It remains to see that the extension just defined satisfies the property claimed in the lemma. Let $v = \sum_{i=0}^n v_i \in V^{k+1}$, $v_i \in V_i^{k+1}$. If $d(v) = d(\alpha)$, for some $\alpha \in \mathcal{M}_k$, then $d(v_n) = d(\alpha - \sum_{i=1}^{n-1} v_i) \in d(\wedge(V^{\leq k} \oplus V_{< n}^{k+1}))$. This forces $v_n = 0$, if $n > 0$. A downward induction on n shows $v = v_0$. \square

Proposition 3.6. *A k -formal space M with $H^{\geq k+2}(M) = 0$ is formal.*

Proof. We use a result of [5] to deduce the formality of M , corresponding to the case $k = \infty$ from our Proposition 3.1, Part (1). That is, we have to find a minimal model $\mathcal{M} = (\wedge V, d)$ that admits a decomposition $V^i = C^i \oplus N^i$ for any i , such that $C^i = \text{Ker}(d|_{V^i})$ and any closed element in the ideal generated by $\bigoplus N^i$, denoted $I(\bigoplus_{i \geq 1} N^i)$, is exact in \mathcal{M} .

The k -formality is characterized on the other hand by Proposition 3.1, Part (1), hence we choose $\mathcal{M}_k = (\wedge V^{\leq k}, d)$ a k -minimal model with a decomposition $V^i = C^i \oplus N^i$ for $i \leq k$ such that any closed element of degree at most $k+1$ in $N^{\leq k} \cdot \mathcal{M}_k$ is exact in \mathcal{M}_k . Extend \mathcal{M}_k to a minimal model \mathcal{M} , such that the \mathcal{M}_{k+1} extension is constructed as in Lemma 3.5.

For $i \geq k+2$ take $\alpha \in V^i$, $d(\alpha) = 0$. Since $H^{\geq k+2}(\mathcal{M}) = 0$, there is $z \in \mathcal{M}$, such that $\alpha = d(z)$. Due to the decomposability of the differential of the minimal algebra \mathcal{M} , $\alpha = 0$. Therefore $C^{\geq k+2} = 0$ and we may take $N^{\geq k+2} = V^{\geq k+2}$.

For $i = k+1$, we know from Lemma 3.5 that $C^{k+1} = V_0^{k+1}$. Take $N^{k+1} = \bigoplus_{i>0} V_i^{k+1}$.

It remains to see that every closed homogeneous element $\alpha \in I(\bigoplus_{i \geq 1} N^i)$, is exact in \mathcal{M} . If α has degree $\geq k+2$, this is clear, since $H^{\geq k+2}(\mathcal{M}) = 0$. If α is of degree $\leq k$, then $\alpha \in N^{\leq k} \cdot \mathcal{M}_k$ and α must be exact in \mathcal{M}_k , hence in \mathcal{M} , by Proposition 3.1, Part (1).

The last case to be solved is when α is homogeneous of degree $k+1$. In this case, $\alpha = \alpha_1 + \alpha_2$, with $\alpha_1 \in (N^{\leq k} \cdot \mathcal{M}_k)^{k+1}$ and $\alpha_2 \in N^{k+1}$. Then $d(\alpha) = 0$ implies $d(\alpha_2) = d(-\alpha_1)$, hence $\alpha_2 = 0$, by Lemma 3.5. The exactness of $\alpha = \alpha_1$ follows again by Proposition 3.1 (1). \square

We remark that [10, Lemma 2.10] is a consequence of the above result.

Corollary 3.7. *Complex plane projective curve complements are formal spaces.*

Proof. One knows that the complements of plane projective curves are 1-formal spaces, having the homotopy type of a CW-complex with cells of dimension ≤ 2 ; see [14] and [6] respectively, for more details. Hence their cohomology is trivial in dimension ≥ 3 and we can apply Proposition 3.6. \square

The same result was proved in [4], using a different approach.

4. PARTIAL FORMALITY AND RESONANCE

Definition 4.1. Let M be a (finite type) connected CW complex. Define the resonance variety $\mathcal{R}_k^q(M)$ as the subset (homogeneous subvariety) of all cohomology classes $w \in H^1(M, \mathbb{k})$ such that $\dim_{\mathbb{k}} H^q(H^*(M), \mu_w) \geq k$, where μ_w is the differential induced by the multiplication by w .

To obtain another type of obstruction to formality, related to resonance varieties, we begin with a lemma on bigraded minimal models.

Lemma 4.2. *Let H^* be a connected graded-commutative algebra and denote by \mathcal{B} its bigraded minimal model as defined and constructed in [11]. If $\mathcal{R}_1^q(H^*) \not\subseteq \{0\}$, then the vector space \mathcal{B}^q has infinite dimension.*

Proof. We evaluate first the resonance varieties of $H^* \cong H^*(\mathcal{B})$.

Take $[\omega] \in H^1(\mathcal{B})$ and $[\eta] \in H^q(\mathcal{B})$ with $d(\omega) = 0$, $d(\eta) = 0$ and $[\omega\eta] = 0$ in $H^{q+1}(\mathcal{B})$. Note that $\omega \in Z_0^1$: write $\omega = \sum_{i \geq 0} \omega_i$, $\omega_i \in Z_i^1$, with almost all $\omega_i = 0$. Now $d(\omega) = 0$ implies $d(\omega_i) = 0$, $\forall i$; moreover, for $i > 0$, all ω_i are boundaries, as follows from (3.1), so $\omega_i = 0$, by minimality. The same type of argument shows that we can take $\eta \in \mathcal{B}_0^q$.

The equality $[\omega\eta] = 0$ in H^{q+1} translates into $\omega\eta = d(\alpha)$ in \mathcal{B}^{q+1} , for some $\alpha \in \mathcal{B}_1^q$. Now suppose $\omega \neq 0$ and $[\eta] \notin [\omega]H^{q-1}(\mathcal{B})$.

If $\alpha = 0$, then $\omega\eta = 0$ in the free graded commutative algebra \mathcal{B} , hence $\eta = \bar{\eta}\omega$ with $\bar{\eta}$ of lower degree 0, hence $d(\bar{\eta}) = 0$. But this implies $[\eta] = \pm[\omega][\bar{\eta}]$, a contradiction.

So, $\alpha := \alpha_1 \neq 0$. We construct a sequence $(\alpha_i)_{i \geq 0}$, $\alpha_i \in \mathcal{B}_i^q$, $\alpha_0 := \eta$, such that $d(\alpha_{i+1}) = \omega\alpha_i$ and $\alpha_i \neq 0 \forall i$. Assume α_n already defined and satisfying the required properties for some fixed $n \geq 1$. Then $d(\omega\alpha_n) = 0$, so there is $\alpha_{n+1} \in \mathcal{B}_{n+1}^q$ such that $d(\alpha_{n+1}) = \omega\alpha_n$. Also $\alpha_{n+1} = 0$ implies $\omega\alpha_n = 0$.

The last equality will lead to a contradiction. If $\omega\alpha_n = 0$, then $\alpha_n = \omega\beta$, $\beta \in \mathcal{B}_n^{q-1}$. Differentiating the last equality we obtain $\omega\alpha_{n-1} = \pm\omega d(\beta)$, hence $\alpha_{n-1} \pm d(\beta) = \omega\zeta_{n-1}$, with $\zeta_{n-1} \in \mathcal{B}_{n-1}^{q-1}$. Differentiating again we get $\omega\alpha_{n-2} = \pm\omega d(\zeta_{n-1})$, which implies $\alpha_{n-2} \pm d(\zeta_{n-1}) = \omega\zeta_{n-2}$ with $\zeta_{n-2} \in \mathcal{B}_{n-2}^{q-1}$. A downward induction shows that eventually we obtain the equality $\eta \pm d(\zeta_1) = \omega\zeta_0$, with $\zeta_1 \in \mathcal{B}_1^{q-1}$ and $\zeta_0 \in \mathcal{B}_0^{q-1}$, which implies $[\eta] = [\omega][\zeta_0]$, again a contradiction.

This ends the construction of the sequence $(\alpha_i)_{i \geq 0}$, $\alpha_i \in \mathcal{B}_i^q$, with nonzero elements α_i and the conclusion follows. \square

Proof of Theorem 1.3 Let us denote by \mathcal{B} the bigraded minimal model of the cohomology algebra $H^*(G)$, and by ${}_s\mathcal{B}$ the s -minimal model of $H^*(G)$. According to the previous lemma, it is enough to show that \mathcal{B}^q has finite dimension, for $q \leq s$. The s -formality of G implies that the s -minimal model of G coincides with the s -minimal model of $H^*(G)$, that is $\mathcal{M}_s(G) = {}_s\mathcal{B}$. Since G is finitely generated and nilpotent, $\mathcal{M}(G) = \mathcal{M}_1(G) = \mathcal{M}_s(G)$ is a finite dimensional vector space. Clearly, $\mathcal{B}^q = {}_s\mathcal{B}^q$, for $q \leq s$. Our proof is complete. \square

Example 4.3. The nilpotency condition in Theorem 1.3 is necessary. We consider $G_{\mathcal{A}} = \pi_1(M_{\mathcal{A}})$ to be the fundamental group of the complement of a central complex hyperplane arrangement $\mathcal{A} \subset \mathbb{C}^n$, $n \geq 3$. Then $G_{\mathcal{A}}$ is finitely generated (see for

instance [17]) and 1-formal ($M_{\mathcal{A}}$ is a formal space -see for example [17]- hence 1-formal, so $G_{\mathcal{A}}$ is also 1-formal).

In this setting, the properties below are equivalent:

- (1) The hyperplanes of \mathcal{A} are in general position in codimension 2.
- (2) The group $G_{\mathcal{A}}$ is abelian.
- (3) The group $G_{\mathcal{A}}$ is nilpotent.
- (4) $\dim_{\mathbb{Q}} \operatorname{gr}(G_{\mathcal{A}}) \otimes \mathbb{Q} < \infty$.
- (5) $\mathcal{R}_1^1(G_{\mathcal{A}}) \subseteq \{0\}$.
- (6) $\mathcal{V}_1^1(G_{\mathcal{A}}) \subseteq \{1\}$.

(Here $\operatorname{gr}(G) \otimes \mathbb{Q}$ is the rational associated graded Lie algebra of a group G , and $\mathcal{V}_1^1(G)$ denotes its (first) characteristic variety in degree one.)

The implication (1) \Rightarrow (2) follows from Hattori's Theorem from [13], and (2) \Rightarrow (3) \Rightarrow (4) are obvious. For (4) \Rightarrow (1), we refer to [8, Proposition 2.12]. The implication (3) \Rightarrow (5) is given by our Theorem 1.3 and (5) \Rightarrow (1) is implicit in the proof of Proposition 2.12, from [8]. (3) \Rightarrow (6) follows from [15, Theorem 1.1]. Finally, (6) \Rightarrow (5), since in the 1-formal case (which is the case for $G_{\mathcal{A}}$) we have a local isomorphism $\mathcal{R}_1^1 \cong \mathcal{V}_1^1$ given by the exponential map; see [7, Theorem A].

Remark 4.4. The partial formality in the hypothesis of Theorem 1.3 is also essential. The more precise claim is that for any finite connected CW-complex M , there is a finitely generated 2-step nilpotent group G such that $\mathcal{R}_k^1(G) = \mathcal{R}_k^1(M)$, for all k . See [15]. Now take $M = M_{\mathcal{A}}$, where the arrangement \mathcal{A} is not in general position in codimension 2. Since $\mathcal{R}_*^1(G_{\mathcal{A}}) = \mathcal{R}_*^1(M_{\mathcal{A}})$, by Definition 4.1, it follows from Example 4.3 that $\mathcal{R}_1^1(G) \not\subseteq \{0\}$.

Notice that the case of resonance varieties of finitely generated nilpotent groups (even restricted to 2-step nilpotent groups) is very different from the one of characteristic varieties, described in [15].

Given a graded-commutative algebra H^* , let K be the kernel of the multiplication map $\mu : H^1 \wedge H^1 \longrightarrow H^2$, called in [3] the characteristic subspace of H^* .

Lemma 4.5. *The subspace K contains no nontrivial decomposables if and only if*

$$(4.1) \quad \{\omega \in K \mid \omega^2 = 0 \in \wedge^4 H^1\} \subseteq \{0\}$$

Both properties are equivalent to $\mathcal{R}_1^1(H^) \subseteq \{0\}$.*

Proof. Assume there is $0 \neq \omega = \sum_{i=1}^m x_i y_i$, written in canonical form, satisfying $\omega^2 = 0$. Since $\omega^i = i! \sum_{1 \leq k_1 < \dots < k_i \leq m} x_{k_1} y_{k_1} \dots x_{k_i} y_{k_i} \neq 0$ if $i \leq m$ we conclude that $m + 1 = 2$, that is ω is a nontrivial decomposable.

The converse implication is immediate: a decomposable element $\omega = \alpha_1 \alpha_2 \neq 0$ satisfies $\omega^2 = 0$.

It is equally easy to see that the fact that K does not contain nontrivial decomposables is equivalent to $\mathcal{R}_1^1(H^*) \subseteq \{0\}$:

Assume there is $\alpha \in \mathcal{R}_1^1(H^*)$, $\alpha \neq 0$, hence there exists $\beta \in H^1$, $\alpha \wedge \beta \neq 0$ such that $\alpha\beta = 0$ in H^2 . This implies $\alpha \wedge \beta$ is a nontrivial decomposable element of K .

On the other hand, if for some $\alpha, \beta \in H^1$ we have $0 \neq \alpha \wedge \beta \in K$, then $\alpha\beta = 0$ in H^2 , hence $\mathcal{R}_1^1(H^*)$ contains nonzero elements. \square

We see in the next Example that the triviality of the resonance varieties, up to a degree, is not a sufficient condition for partial formality, even when the cohomology is partially generated in degree one.

Example 4.6. Let $\mathcal{B} = \wedge(x_1, x_2, y_1, y_2, z, \omega_1, \omega_2, \alpha)$, $d(x_i) = d(y_i) = d(z) = 0$, $d(\omega_1) = x_1 y_1 + x_2 z$, $d(\omega_2) = x_2 y_2 + x_1 z$, $d(\alpha) = x_1 \omega_1 + x_2 \omega_2$, be the 1-formal minimal model generated in degree 1 from [3], Example 2.8.

Then \mathcal{B} , in the notations from the beginning of Section 3, admits a bigrading compatible with d , with $Z_0 = \langle x_i, y_i, z \rangle$, $Z_1 = \langle \omega_1, \omega_2 \rangle$, $Z_2 = \langle \alpha \rangle$. One can check that $H_+^{\leq 2}(\mathcal{B}) = 0$. Consider now a "deformation" of \mathcal{B} , $\mathcal{M} = \wedge(Z_0 \oplus Z_1 \oplus Z_2)$, with differential D defined as follows: $D|_{Z_0, Z_1} = d|_{Z_0, Z_1}$ and $D(\alpha) = d(\alpha) + p$, $p \in \wedge^2 Z_0$. A direct computation shows that $D^2 = 0$.

Next we will prove that $H^{\leq 2} \mathcal{M} \cong H^{\leq 2} \mathcal{B}$, as algebras. As explained in [19], \mathcal{B} is the minimal model of a finitely presentable (3-step) nilpotent group $G_{\mathcal{B}}$. Hence, via Proposition 3.1, Part (3) and Theorem 1.3, applied for $M = K(G_{\mathcal{B}}, 1)$, one finds that $H^{\leq 2} \mathcal{M}$ is generated in degree 1 and $\mathcal{R}_1^1(\mathcal{M}) \subseteq \{0\}$.

Obviously $H^1 \mathcal{M} = H^1 \mathcal{B} = Z_0$. Take $w = \bar{w} + \xi \alpha$ such that $d(w) = 0$, $\bar{w} \in \wedge^2(Z_0 \oplus Z_1)$ and $\xi \in \wedge^1(Z_0 \oplus Z_1)$. Then $d(w) = d(\bar{w}) + d(\xi)\alpha - \xi d(\alpha) = 0$ implies $d(\xi) = 0$, hence $\xi \in \wedge^1(x_i, y_i, z)$. It follows that

$$(4.2) \quad d(\bar{w}) - \xi x_1 \omega_1 - \xi x_2 \omega_2 = 0.$$

Notice that there are monomials in $d(\bar{w})$ containing ω_i if and only if \bar{w} has a monomial $a\omega_1\omega_2$ with $a \neq 0$; grouping the monomials in (4.2) which contain ω_1 (respectively ω_2), we conclude that $\xi x_1 = 0$ and $\xi x_2 = 0$, which is possible only when $\xi = 0$.

So $d(w) = 0$ implies $w \in \wedge^2(x_i, y_i, z, \omega_i)$. The same way $D(w) = 0$ implies $w \in \wedge^2(x_i, y_i, z, \omega_i)$, hence $d(w) = 0 \Leftrightarrow D(w) = 0$, so w is a cocycle in \mathcal{B} if and only if w is a cocycle in \mathcal{M} . Since $\text{im } d|_{\mathcal{B}^1} \cong \text{im } D|_{\mathcal{M}^1}$, we obtain a vector space isomorphism:

$$(4.3) \quad H^{\leq 2} \mathcal{M} \cong H^{\leq 2} \mathcal{B}$$

We still need a graded algebra isomorphism. Using the previous notations, consider the graded algebra $\mathcal{C}^* := \frac{\wedge^* Z_0}{\wedge^* Z_0 \cdot dZ_1}$. Notice that we have $H^{\leq 2} \mathcal{B} \cong \mathcal{C}^{\leq 2}$, as algebras, since $H_+^{\leq 2}(\mathcal{B}) = 0$. Define a graded algebra morphism

$$(4.4) \quad \psi : \mathcal{C}^* \longrightarrow H^*(\mathcal{M})$$

given by $\psi(z) = [z]$, $\forall z \in Z_0$. It is clear that ψ^1 is a linear isomorphism, and ψ^2 is a surjection between vector spaces of the same dimension, according to the above computations (see (4.3)).

However, we will see that the 1-minimal model \mathcal{M} is not 1-formal, if $p = y_1 y_2$. Assuming the contrary, we have an isomorphism of D.G. algebras $\phi : \mathcal{B} \longrightarrow \mathcal{M}$ (since \mathcal{B} is the 1-minimal model of the cohomology algebra of \mathcal{M}).

Moreover, we can choose ϕ such that $\phi|_{Z_0} = \text{id}$. Indeed, let $\tilde{h} : \mathcal{B} \xrightarrow{\sim} \mathcal{B}$ be the 1-minimal model of the graded algebra automorphism induced by ϕ , $h : H^{\leq 2}(\mathcal{B}) \xrightarrow{\sim} H^{\leq 2}(\mathcal{M}) \equiv H^{\leq 2}(\mathcal{B})$. Replacing an arbitrary ϕ by $\phi \circ \tilde{h}^{-1}$, we obtain the desired property.

Checking the equality $D\phi = \phi d$ on the generators ω_i , one gets $\phi(\omega_i) - \omega_i \in \wedge^1(x_i, y_i, z)$. Clearly, $\phi(d(\alpha))$ is a sum of monomials, each containing either x_1 or x_2 . Let $\phi(\alpha) = a_1 x_1 + a_2 x_2 + b_1 y_1 + b_2 y_2 + cz + d_1 \omega_1 + d_2 \omega_2 + e\alpha$, with $a_i, b_i, c, d_i, e \in \mathbb{k}$. Notice that $e \neq 0$, otherwise $\phi : \mathcal{B}^1 \rightarrow \mathcal{M}^1$ would not be a linear isomorphism. Then $D(\phi(\alpha))$ necessarily contains the monomial $y_1 y_2$, with nontrivial coefficient e . But this contradicts the fact that ϕ is a D.G.A. morphism.

Remark 4.7. Let G be a finitely presentable group, with 1-minimal model \mathcal{M} . In Lemma 3.17 on page 35, implication (i) \Rightarrow (ii), the authors of [1] note that $H^2(\mathcal{M}) = (H^1(\mathcal{M}))^2$, if G is 1-formal. This can be recovered from our Proposition 3.1(1)-(2), case $k = 1$.

On the other hand, this implication cannot be reversed, contrary to the claim from [1, Lemma 3.17]. Indeed, the 1-minimal model \mathcal{M} constructed in Example 4.6 above can be realized as $\mathcal{M} = \mathcal{M}(G) = \mathcal{M}_1(G)$, where G is a finitely presentable (3-step) nilpotent group, by the general theory from [19]. It follows from Example 4.6 that $H^2(\mathcal{M}) = (H^1(\mathcal{M}))^2$, yet G is not 1-formal.

5. HEISENBERG-TYPE GROUPS

Definition 5.1. The integral Heisenberg group \mathcal{H}_n is given by the central extension

$$(5.1) \quad 0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{H}_n \longrightarrow \mathbb{Z}^{2n} \longrightarrow 0,$$

corresponding to the cohomology class $\omega \in H^2(\mathbb{Z}^{2n}, \mathbb{Z}) = \wedge_{\mathbb{Z}}^2(x_1, y_1, \dots, x_n, y_n)$, where $\omega = x_1 \wedge y_1 + \dots + x_n \wedge y_n$.

It is immediate that the minimal model of \mathcal{H}_n is the minimal DG algebra generated in degree 1, $\mathcal{M} = \wedge(x_1, y_1, \dots, x_n, y_n, z)$, with differential $d(x_i) = d(y_i) = 0, \forall i$ and $d(z) = x_1 \wedge y_1 + \dots + x_n \wedge y_n$. Note that the multiplication by ω in the exterior algebra $E^* = \wedge^*(x_1, y_1, \dots, x_n, y_n)$, $E^i \xrightarrow{\mu_\omega} E^{i+2}$, is injective for $i \leq n-1$, by the hard Lefschetz theorem, see [20].

Lemma 5.2. *The cohomology of the Heisenberg group \mathcal{H}_n is given by*

$$(5.2) \quad H^q(\mathcal{H}_n) \cong \frac{\wedge^q(x_i, y_i)}{\omega \wedge^{q-2}(x_i, y_i)} \oplus \{\eta z \mid \eta \omega = 0, \eta \in \wedge^{q-1}(x_i, y_i)\}, \forall q.$$

The second summand is trivial, for $q \leq n$, and non-trivial, for $q = n+1$.

Proof. It is clear that

$$(5.3) \quad H^1(\mathcal{H}_n) = \wedge^1(x_i, y_i)$$

Let us compute $H^q(\mathcal{H}_n)$, for $2 \leq q$. Any q -form $\xi \in \wedge^q(x_i, y_i, z)$ is of the type $\xi = \eta_1 + \eta_2 z$, where $\eta_1 \in \wedge^q(x_i, y_i)$ and $\eta_2 \in \wedge^{q-1}(x_i, y_i)$. Hence ξ is a cocycle if and only if $\pm d(\xi) = \eta_2 \omega = 0$. In case $q \leq n$ the last equality implies $\eta_2 = 0$. Moreover we get that any q -coboundary is of the type $\eta_2 \omega$, $\eta_2 \in \wedge^{q-2}(x_i, y_i)$. Consequently, $H^q(\mathcal{H}_n)$ has the asserted form. Clearly, $\eta = y_1 \cdots y_n$ creates a nontrivial contribution of the second summand, in degree $n+1$. \square

Remark 5.3. The above Lemma shows that \mathcal{H}_n is $(n-1)$ -formal (use Theorem 1.2, Part (2)), but not n -formal (as follows from Theorem 1.2, Part (1)). At the same time, it shows that the notion of partial formality from [10] is strictly stronger than the one used here.

Consider the 1-formal (in the sense of our Definition 2.5) group \mathcal{H}_2 . The 1-formality of \mathcal{H}_2 in the sense of [10, Definition 2.2] would imply a decomposition of the space of degree 1 generators $\langle x_1, x_2, y_1, y_2, z \rangle$ as a direct sum $C^1 \oplus N^1$, satisfying the conditions:

- (i) $d(C^1) = 0$;
- (ii) the restriction of the differential d to N^1 is injective;
- (iii) any closed element in the ideal generated by N^1 in $\mathcal{M} = \mathcal{M}_1(\mathcal{H}_2) = \mathcal{M}(\mathcal{H}_2)$ is exact in $\mathcal{M}(\mathcal{H}_2)$.

In this case C^1 is the subspace $\langle x_1, x_2, y_1, y_2 \rangle$ and N^1 must be generated by $z + \alpha$ with $\alpha \in \wedge^1(C^1)$. Plainly, $(z + \alpha)x_1x_2y_1y_2$ is closed, but not exact.

Let us compute some resonance varieties for Heisenberg groups:

Proposition 5.4. $\mathcal{R}_1^*(\mathcal{H}_n) = \{0\}$, for $* \leq n-1$ and $\mathcal{R}_1^n(\mathcal{H}_n) = \mathbb{k}^{2n}$.

Proof. We know the additive structure of $H^*(\mathcal{H}_n)$ (see Lemma 5.2). We have to compute the cohomology of the complex $H^*(\mathcal{H}_n)$ with differential given by the multiplication with the class of an element $0 \neq \xi \in \wedge^1(x_i, y_i)$.

We may assume $\xi = x_1$, by a linear change of coordinates. The n -class $[y_1 \dots y_n]$ cannot be obtained by multiplying some $(n-1)$ -class by $[x_1]$: the equality $y_1 \dots y_n = x_1 \eta + \omega \beta$, with $\eta \in \wedge^{n-1}(x_i, y_i)$ and $\beta \in \wedge^{n-2}(x_i, y_i)$ is impossible, since at the right hand side all monomials contain some x_i component.

However $x_1 y_1 \dots y_n = \omega y_2 \dots y_n$, hence $[x_1 y_1 \dots y_n] = 0$ in $H^{n+1}(\mathcal{H}_n)$; so $[x_1] \in \mathcal{R}_1^n(\mathcal{H}_n)$. This proves that $\mathcal{R}_1^n(\mathcal{H}_n) = \wedge^1(x_i, y_i)$.

We can apply Theorem 1.3 and deduce $\mathcal{R}_1^*(\mathcal{H}_n) \subseteq \{0\}$, for $* < n$.

It remains to see that $0 \in \mathcal{R}_1^*(\mathcal{H}_n)$, if $* < n$. Assuming the contrary, it follows from Lemma 5.2 that $H^n(\mathcal{H}_n) = 0$. But this implies $\mathcal{R}_1^n(\mathcal{H}_n) = \emptyset$, contradicting the first computation. \square

However, the triviality of the resonance varieties and the property of having the cohomology generated in degree 1 are independent in general, as we can see below:

Example 5.5. Let G be a finitely generated 2-step nilpotent group with minimal model generated in degree 1, $\mathcal{N} = \wedge(x_1, x_2, y_1, y_2, z) \otimes \wedge(\omega_1, \omega_2)$ with differentials $dz = dx_i = dy_i = 0$, $i = 1, 2$ and $d\omega_1 = x_1 y_1 + x_2 z$, $d\omega_2 = x_2 y_2 + x_1 z$; see Example 4.6. Then $\mathcal{R}_1^1(G) = \{0\}$, but $H^2(G) \neq (H^1(G))^2$.

Set $K := K(G, 1)$. First we compute the cohomology of K in low degrees. It is immediate that $H^1(K) = \wedge^1(x_i, y_i, z)$ and $H^2(K) = H_0^2(K) \oplus H_1^2(K) \oplus H_2^2(K)$, where $H_0^2(K) = \frac{\wedge^2(x_i, y_i, z)}{\langle d\omega_1, d\omega_2 \rangle}$. A direct computation shows that $H_1^2(K) = \langle x_1 \omega_1 + x_2 \omega_2 \rangle$, and $H_2^2(K) = 0$.

Hence $H^2(K) = \frac{\wedge^2(x_i, y_i, z)}{\langle d\omega_1, d\omega_2 \rangle} \oplus \langle x_1 \omega_1 + x_2 \omega_2 \rangle \neq (H^1(K))^2$.

As for the resonance variety $\mathcal{R}_1^1(K)$, take a one-cycle $\xi \neq 0$ and a one-cycle η such that $[\eta\xi] = 0$ in cohomology. This can happen only if $\eta\xi = ad\omega_1 + bd\omega_2$ for some $a, b \in \mathbb{k}$.

If $\eta\xi = ad\omega_1 + bd\omega_2$, then $0 = (\eta\xi)^2 = a^2(d\omega_1)^2 + 2abd\omega_1 d\omega_2 + b^2(d\omega_2)^2$, i.e. $0 = 2a^2 x_1 y_1 x_2 z + 2ab x_1 y_1 x_2 y_2 + 2b^2 x_2 y_2 x_1 z$, so $a = b = 0$. Consequently $\eta\xi = 0$ in $\wedge(x_1, x_2, y_1, y_2, z)$. Since $\xi \neq 0$, $\eta \in \langle \xi \rangle$.

Therefore, $\mathcal{R}_1^1(K) = \{0\}$.

Let G be the 2-step nilpotent group defined by the central extension:

$$(5.4) \quad 0 \longrightarrow B \longrightarrow G \longrightarrow A \longrightarrow 0,$$

where B is an abelian group of rank 1, and A is an abelian group of finite rank n .

The minimal model of G from (5.4) is of the form $\mathcal{M}(G) = \wedge(t_1, \dots, t_n) \otimes \wedge(z)$, with differentials $d(t_i) = 0, \forall i$ and $d(z) := \omega \in \wedge^2(t_1, \dots, t_n)$.

Definition 5.6. G is called of Heisenberg-type if $\omega \neq 0$.

Moreover we may assume ω has the canonical form $\omega = x_1 y_1 + \dots + x_m y_m$, where $2m = \text{rk}(\omega)$; consequently:

$$(5.5) \quad \mathcal{M}(G) = \mathcal{M}(\mathcal{H}_m) \otimes (\wedge(t_{2m+1}, \dots, t_n), d = 0).$$

To obtain information on the resonance varieties associated to Heisenberg-type groups, we will use the following result:

Proposition 5.7. *Let A^*, B^* be graded-commutative connected algebras. Then*

$$(5.6) \quad \mathcal{R}_1^q(A^* \otimes B^*) = \bigcup_{m+n=q} \mathcal{R}_1^m(A^*) \times \mathcal{R}_1^n(B^*)$$

Proof. Set $C^* = A^* \otimes B^*$. If $\xi = \xi_A + \xi_B \in C^1 = A^1 \oplus B^1$ is an arbitrary degree 1 element, then the multiplication by ξ on C^* is given by:

$$(5.7) \quad \mu_\xi(a \otimes b) = \mu_{\xi_A}(a) \otimes b + (-1)^{|a|} a \otimes \mu_{\xi_B}(b), \quad a \in A^*, \quad b \in B^*.$$

Consequently, by Künneth,

$$(5.8) \quad H^q(C^*, \mu_\xi) = \bigoplus_{m+n=q} H^m(A^*, \mu_{\xi_A}) \otimes H^n(B^*, \mu_{\xi_B}).$$

By definition, $\mathcal{R}_1^q(A^* \otimes B^*) = \{\xi \in A^1 \oplus B^1 \mid H^q(A^* \otimes B^*, \mu_\xi) \neq 0\} = \{(\xi_A, \xi_B) \in A^1 \times B^1 \mid \bigoplus_{m+n=q} H^m(A^*, \mu_{\xi_A}) \otimes H^n(B^*, \mu_{\xi_B}) \neq 0\} = \{(\xi_A, \xi_B) \in A^1 \times B^1 \mid \exists(m, n), m+n=q, H^m(A^*, \mu_{\xi_A}) \neq 0 \text{ \& } H^n(B^*, \mu_{\xi_B}) \neq 0\} = \{(\xi_A, \xi_B) \in A^1 \times B^1 \mid \exists(m, n), m+n=q, \xi_A \in \mathcal{R}_1^m(A^*) \text{ \& } \xi_B \in \mathcal{R}_1^n(B^*)\} = \bigcup_{m+n=q} \mathcal{R}_1^m(A^*) \times \mathcal{R}_1^n(B^*)$. \square

Corollary 5.8. *Let G be a Heisenberg-type group, with minimal model described in (5.5). Then $\mathcal{R}_1^*(G) = \{0\}$, for $* \leq m-1$ and $\mathcal{R}_1^m(G) = \mathbb{K}^{2m}$.*

Proof. By (5.5) and Proposition 5.7, $\mathcal{R}_1^k(G)$ equals

$$\mathcal{R}_1^k(H^*(\mathcal{H}_m) \otimes \wedge^*(t_{2m+1}, \dots, t_n)) = \bigcup_{p+q=k} \mathcal{R}_1^p(\mathcal{H}_m) \times \mathcal{R}_1^q(\wedge^*(t_{2m+1}, \dots, t_n)).$$

Since the resonance for the exterior algebra is trivial, the corollary follows from Proposition 5.4. \square

Corollary 5.9. *A Heisenberg-type group G with $\text{rk}(\omega) = 2m$ is $(m-1)$ -formal, but not m -formal.*

Proof. One can easily see from Theorem 1.3, via Corollary 5.8, that G is not m -formal. The $(m - 1)$ -formality follows from Theorem 1.2, Part (2) and Lemma 5.2, using again (5.5) to describe the cohomology ring of G up to degree m . \square

Corollary 5.10. *A Heisenberg-type group G with $\mathrm{rk}(\omega) = 2m$ cannot be realized as the fundamental group of a smooth projective complex variety M with $\pi_{\leq m}(\tilde{M}) = 0$, where \tilde{M} is the universal covering of M .*

Proof. Assume $G = \pi_1(M)$, with M a smooth projective complex variety. By the main result of [5], M is a formal space, hence m -formal, while G is not m -formal (see Corollary 5.9). Then there is $2 \leq i \leq m$ such that $\pi_i(\tilde{M}) \cong \pi_i(M) \neq 0$; see Theorem 1.1. \square

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